Adaptive Design and Recursive Estimation

STAT527 Final Project

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Outline

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1. Basics of adaptive design
2. Techniques in the literature
3. (Potential) applications
4. Basics of recursive estimation
5. Simulations

Adaptive design (Lai and Robbins, 1978), or stochastic regression (Lai and Wei, 1982), considers the model

$$y_i = M(x_i) + \epsilon_i, \tag{1}$$

where for $i = 1, 2, \ldots,$

- $\{x_i\}$ **is sequentially determined**;
- $M(\cdot)$ may be nonlinear;
- $\{\epsilon_i\}$ may be iid or martingale difference sequence (mds).

Meaning of adaptive design

In fixed and random design, $\{x_i\}$ is deterministic and random, respectively. However, we may not have control over $\{x_i\}$, whereas we do in adaptive design. Interestingly, there is another term "design-adaptive"; see Fan (1992).

Lai and Robbins (1978) motivated adaptive design with the following setting:

- y_i: response value of the i-th patient;
- x_i: dosage level of some drug;
- y^*: optimal response value, which is known;
- $M(\cdot)$: regression function, which is unknown;
- θ: optimal dosage level, which is unknown as we cannot solve $y^* = M(\theta)$.

A multi-period control problem under uncertainty

How to choose $\{x_i\}$ sequentially such that $\{y_i\}$ is as close as possible in some sense to y^*? There is a dilemma between a good final estimate of θ and a small cost, e.g., $\sum_{i=1}^{n}(x_i - \theta)^2$.

Simplified problem, known β

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For presentation purpose, assume $\epsilon_i \overset{\text{iid}}{\sim} \mathrm{N}(0, \sigma^2)$, and

$$M(x_i) = \beta(x_i - \theta), \tag{2}$$

which corresponds to $y^* = M(\theta) = 0$. Let the cost at stage n be

$$C_n = \sum_{i=1}^{n}(x_i - \theta)^2. \tag{3}$$

To estimate θ when β is known, we may use the least squares estimator

$$\hat{\theta}_n = \bar{x}_n - \bar{y}_n/\beta = \theta - \bar{\epsilon}_n/\beta, \tag{4}$$

where $\bar{x}_n = \sum_{i=1}^{n} x_i/n$.

To minimize C_n, adaptive design let x_1, the initial best guess of θ, to be a random variable such that $\mathbb{E}(x_1^2) < \infty$, and

$$x_{n+1} = \hat{\theta}_n = \theta - \bar{\epsilon}_n/\beta. \tag{5}$$

An advantage of adaptive design

Adaptive design reduces the expected cost from $O(n)$ to $O(\log n)$.

Proof:

$$\begin{aligned}\mathbb{E}(C_n) &= \mathbb{E}(x_1 - \theta)^2 + \sum_{i=1}^{n-1} \mathbb{E}(\theta_i - \theta)^2 \\ &= O(1) + \frac{\sigma^2}{\beta}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) \sim \frac{\sigma^2}{\beta} \log n.\end{aligned}$$

Assuming a simple linear M and a known β are not realistic. There are different extensions, which still keep the advantages of adaptive design in general:

1. Univariate nonlinear M under some conditions (Lai and Robbins, 1979);
2. Multivariate linear M (Lai and Wei, 1982);
3. Fixed guess of β (Lai and Robbins, 1979);
4. Stochastic approximated estimates of β (Lai and Robbins, 1979);
5. Iterated least squares estimates of β (Lai and Robbins, 1982).

Theoretical techniques

It is difficult to analyze the theoretical techniques, which are beyond the level of this course. However, we can get to know the them (see Lai and Ying (2006) for their history), which may be helpful for future research:

- Convergence system;
- Lacunary system;
- Extended stochastic Lyapunov function.

Definition

A sequence of random variables ϵ_i is called a convergence system if

$$\sum_{i=1}^{\infty} a_i \epsilon_i \text{ converges a.s.}$$

for all nonrandom $\{a_i\}_{i \geq 1}$ such that $\sum_{i=1}^{\infty} a_i^2 < \infty$. This was used to establish strong consistency of least squares estimates in multiple linear regression with fixed design and mds noise.

Comment

The standard technique to prove almost sure convergence is the Borel–Cantelli lemma. Convergence system provides an alternative way to do so. If ϵ_i's are not iid nor mds, we may consider weakening the dependence, e.g., some mixing condition.

Definition

A sequence of random variables ϵ_i is called a lacunary system of order $p > 0$, or S_p system, if there exists a positive constant K_p such that for all nonrandom a_i,

$$\mathbb{E}\left|\sum_{i=m}^{n} a_i \epsilon_i\right|^p \le K_p \left(\sum_{i=m}^{n} a_i^2\right)^{\frac{p}{2}} \quad \text{for all} \quad n \ge m \ge m_0.$$

This was used to generalize the consistency theorems with fixed design.

Comment

It seems that this technique reduces the order from $O\{(n-m)^p\}$ to $O\{(n-m)^{p/2}\}$. Reduction of this kind reminds us of the Burkholder inequality.

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Definition

Let $\{\epsilon_n, \mathcal{F}_n, n \geq 1\}$ be a martingale difference such that $\sup_n \mathbb{E}(\epsilon_n^2 \mid \mathcal{F}_{n-1}) < \infty$ a.s.. An extended stochastic Lyapunov function V_n is a nonnegative $\mathcal{F}_n$-measurable random variable satisfying

$$V_n \leq (1 + a_{n-1}) V_{n-1} + b_n - c_n + w_{n-1}\epsilon_n \quad \text{a.s.,}$$

where $a_n \geq 0$, $b_n \geq 0$, $c_n \geq 0$ and w_n are $\mathcal{F}_n$-measurable random variables such that $\sum_{n=1}^{\infty} a_n < \infty$. This was used to establish the consistency theorems with adaptive design.

Comment

This seems important in the analysis of stochastic approximation algorithms. Strangely, we do not see it in recent literature.

(Potential) applications

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Adaptive stochastic gradient descent

Estimation of β is similar to selection of learning rate in stochastic gradient descent; see Lai and Robbins (1979) for the relationship between adaptive design and stochastic approximation.

Reinforcement learning and bandit problem

Jingbo has explained the importance of elliptical potential lemma in class. See also Lai and Robbins (1985).

Operation research

If x_i's are prices and y_i's are profits on a e-commerce platform, adaptive design may be used for profit maximization. We need to deal with issues like unknown y^* and changing M though.

In adaptive design, $x_{n+1} = \bar{x}_n - \bar{y}_n/\hat{\beta}_n$ is chosen sequentially. It is trivial that $\bar{y}_n$ and

$$\bar{x}_n = \frac{(n-1)\bar{x}_{n-1} + x_n}{n}, \tag{6}$$

can be updated in constant time. However, usual computation of $\hat{\beta}_n$ costs $O(n)$ time, which motivates recursive estimation:

$$\hat{\beta}_n = \hat{\beta}_{n-1} + P_n x_n (y_n - x_n \hat{\beta}_{n-1}), \tag{7}$$

$$P_n = P_{n-1} - \frac{P_{n-1} x_n x_n P_{n-1}}{1 + x_n P_{n-1} x_n}. \tag{8}$$

Lai and Ying (1991*a*) and Lai and Ying (1991*b*) extended this iterated least squares to other settings.

Usage of recursive estimation

Interestingly, apart from using a similar term "design-adaptive" to refer to a different thing, Fan (1992) also led to a different usage of recursive estimation. In Fan and Marron (1994), recursive estimation technique was used to improve the time complexity for a fixed sample size n; see also Seifert et al. (1994). In contrast, recursive estimators in adaptive design concern an increasing n where the observations arrive sequentially.

Consider

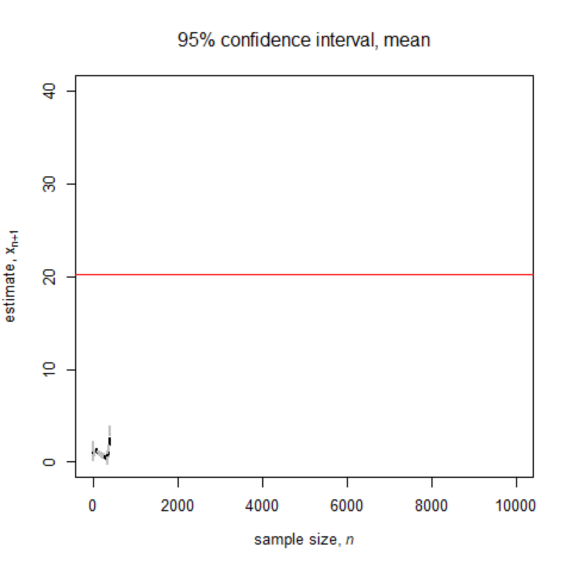
$$y_n = 12.3(x_n - 20.21) + \epsilon_n,$$

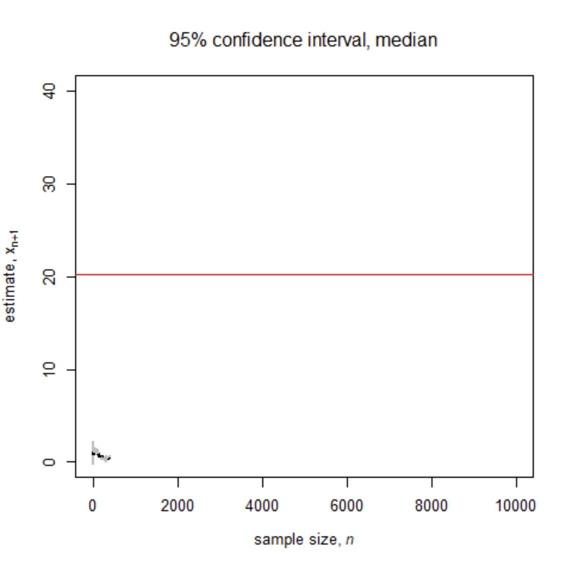
where

- $\epsilon_n \overset{\text{iid}}{\sim} \mathrm{N}(0, 1000^2)$;
- $x_{n+1} = \bar{x}_n - \bar{y}_n / \hat{\beta}_n$;
- $\hat{\beta}_n$ is updated recursively using (7) and (8).

How to do inference on x_{n+1}, or function of $x_1, \ldots, x_n$ in general? We may use the self-normalizer in Shao (2010), which can also be updated recursively.

Simulations, mean of $x_1, \ldots, x_n$




95% confidence interval, median
estimate, x_{n+1}
0
10
20
30
40
0
2000
4000
6000
8000
10000
sample size, n

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